Effective generation of Ising interaction and cluster states in coupled microcavities

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Abstract

We propose a scheme for realizing the Ising spin-spin interaction and atomic cluster states utilizing trapped atoms in coupled microcavities. It is shown that the atoms can interact with each other via the exchange of virtual photons of the cavities. Through suitably tuning the parameters, an effective Ising spin-spin interaction can be generated in this optical system, which is used to produce the cluster states. This scheme does not need the preparation of initial states of atoms and cavity modes, and is insensitive to cavity decay.

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Strongly correlated many-particle systems have been extensively explored in condensed matter physics, cold atoms[1, 2], and recently in optical system such as coupled microcavity lattices[3, 4, 5, 6, 7, 8, 9]. Compared to other strongly correlated many particle systems, an optical system has the advantage of easily addressing individual lattice sites with optical lasers. Because of the size and separation of the microcavities, arbitrary lattice geometries can be arranged in this system. Therefore, it offers the ability to experimentally observe quantum-many-body phenomena and realize quantum information processing. Various technologies have been employed in this field, including fiber coupled toroidal microcavities[10], arrays of defects in photonic band gap materials[11], and superconducting qubits coupled through microwave stripline resonators[12].

On the other hand, interacting qubits are particularly important in quantum information processing. Lattices of interacting spins or qubits can be utilized to generate highly entangled states, such as cluster states[13, 14]. It has been shown that this class of entangled states are much more entangled than the GHZ states, thus have a high persistence of entanglement[13, 14]. The experimental demonstration of the violation of Bell's inequality for cluster states has been reported[15]. Moreover, cluster states together with local measurements are the resource for one-way quantum computation[16, 17]. In the context of cavity QED[18, 19, 20], several proposals have been presented for producing the atomic cluster states[21, 22]. However, the scalable implementation in many-particle case is difficulty. Recently in Ref. [5] a very novel scheme has been proposed to simulate the dynamics of an effective anisotropic Heisenberg spin-1/2 chains and generate atomic cluster states in coupled cavities. However, there is no report of the realization of Ising interaction and cluster states in a simple enough atom-cavity system that does not require the complicated atomic level structure and driving laser configurations.

In this paper we present a scheme for realizing the Ising spin-spin interaction and atomic cluster states utilizing trapped two-level atoms in coupled microcavities arranged in an array. We show that the atoms can interact with each other via the exchange of virtual photons of the cavities. Through suitably tuning the parameters, an effective Ising spin-spin interaction can be generated in this optical system. We discuss how to use this Ising interaction to produce the cluster states of atoms. This scheme does not need the preparation of initial states of atoms and cavity modes, and is insensitive to cavity decay. With presently available experimental setups in cavity QED, the implementation of this scheme is feasible.

As sketched in Fig. 1, an array of microcavities are coupled via the exchange of photons with identical two-level atom in each cavity. The ground state of each atom is labeled as $|g_j\rangle$, and the excited state as $|e_j\rangle$, where the index j counts the cavities. The cavity mode (frequency ω_c) couples to the transition $|g\rangle \leftrightarrow |e\rangle$ (transition frequency ω_0) with coupling constants g. Furthermore, an external strong classical field (frequency ω_L) drives the same transition with Rabi frequencies Ω[23]. For simplicity, we only consider the one-dimensional array. One can generalize to the higher dimensions straightforwardly. In the rotating-wave approximation, the associated Hamiltonian reads (let $\hbar = 1$)

$$\begin{aligned} H &= H_a + H_c + H_{ac}, \\ H_a &= \sum_{j=1}^{N} \omega_0 S_j^+ S_j^- \\ H_c &= \omega_c \sum_{j=1}^{N} \hat{a}_j^\dagger \hat{a}_j + J_c \sum_{j=1}^{N} (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger), \\ H_{ac} &= \sum_{j=1}^{N} [g(\hat{a}_j^\dagger S_j^- + \hat{a}_j S_j^+) + \Omega(S_j^+ e^{-i\omega_L t} + S_j^- e^{i\omega_L t})], \end{aligned} \tag{1}$$

where $S_j^+ = |e_j\rangle\langle g_j|$, $S_j^- = (S_j^+)^\dagger$, $\hat{a}_j$ is annihilation operator for the photon in cavity j, and J_c is the hopping rate of photons between neighboring cavities. In the complete Hamiltonian, H_a describes the free Hamiltonian for atoms; H_c describes the Hamiltonian for photons in each cavity modes, with photon hopping between neighboring cavities; finally H_{ac} is the Hamiltonian that describes the interaction between atoms and the cavities as well as the strong driving by the classical fields. We consider the periodic boundary conditions, then H_c can be diagonalized through the Fourier transformation. For convenience we introduce the notation $\mathbf{J} = (uj, 0, 0)$ to denote the position of the jth site where u is the length of the one-dimensional crystal cell and in the following chosen as unit for simplicity. Then we obtain $H_c = \sum_k \omega_k a_k^\dagger a_k$, where $\omega_k = \omega_c + 2J_c \cos k$.

The Hamiltonian of Eq. (1) can be changed to a reference frame rotating with the driving

field frequency ω_L,

$$\begin{aligned} H = \quad & \Delta \sum_{j=1}^{N} S_j^+ S_j^- - \sum_k \delta_k a_k^\dagger a_k + \sum_{j=1}^{N} \Omega (S_j^+ + S_j^-) \\ & + \sum_{j=1}^{N} [g S_j^- \sum_k a_k^\dagger e^{i\mathbf{k}\cdot\mathbf{J}} + g S_j^+ \sum_k a_k e^{-i\mathbf{k}\cdot\mathbf{J}}], \end{aligned} \tag{2}$$

where $\Delta = \omega_0 - \omega_L$, and $\delta_k = \omega_L - \omega_k$. In the following we assume $\omega_0 = \omega_L$ for simplicity. We now switch to a new atomic basis $|\downarrow_j\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle + |e_j\rangle)$ and $|\uparrow_j\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle - |e_j\rangle)$, then can rewrite H as

$$\begin{aligned} H = \quad & -\sum_k \delta_k a_k^\dagger a_k + \sum_{j=1}^{N} \Omega \sigma_j^z \\ & + \sum_{j=1}^{N} [g(\frac{1}{2}\sigma_j^z + \frac{1}{2}\sigma_j^+ - \frac{1}{2}\sigma_j^-) \sum_k a_k^\dagger e^{i\mathbf{k}\cdot\mathbf{J}} \\ & + g(\frac{1}{2}\sigma_j^z + \frac{1}{2}\sigma_j^- - \frac{1}{2}\sigma_j^+) \sum_k a_k e^{-i\mathbf{k}\cdot\mathbf{J}}]. \end{aligned} \tag{3}$$

where $\sigma_j^z = |\downarrow_j\rangle\langle\downarrow_j| - |\uparrow_j\rangle\langle\uparrow_j|$, $\sigma_j^+ = |\uparrow_j\rangle\langle\downarrow_j|$, and $\sigma_j^- = (\sigma_j^+)^\dagger$. In the interaction picture with respect to $H_0 = -\sum_k \delta_k a_k^\dagger a_k + \sum_{j=1}^{N} \Omega \sigma_j^z$, we have the following interaction Hamiltonian[23]

$$\begin{aligned} H_I = \quad & \sum_{j=1}^{N} [g(\frac{1}{2}\sigma_j^z + \frac{1}{2}\sigma_j^+ e^{-i\Omega t} - \frac{1}{2}\sigma_j^- e^{i\Omega t}) \sum_k a_k^\dagger e^{i\mathbf{k}\cdot\mathbf{J} - i\delta_k t} \\ & + g(\frac{1}{2}\sigma_j^z + \frac{1}{2}\sigma_j^- e^{i\Omega t} - \frac{1}{2}\sigma_j^+ e^{-i\Omega t}) \sum_k a_k e^{-i\mathbf{k}\cdot\mathbf{J} + i\delta_k t}]. \end{aligned} \tag{4}$$

In the strong driving regime $\Omega \gg g, \delta_k$ (for all k), we can realize a rotating-wave approximation and neglect the fast oscillating terms. Then H_I reduces to

$$H_I = \sum_{j=1}^{N} \frac{1}{2} g \sigma_j^z (\sum_k a_k^\dagger e^{i\mathbf{k}\cdot\mathbf{J} - i\delta_k t} + \sum_k a_k e^{-i\mathbf{k}\cdot\mathbf{J} + i\delta_k t}). \tag{5}$$

To further reduce the model, we assume $\delta_k \gg g$ (for all k). Then there is no energy exchange between the atomic system and the cavities. We can adiabatically eliminate the photons from the description[24]. We consider the terms up to second order in the effective

Hamiltonian and drop the fast oscillating terms. Then we obtain the following effective Hamiltonian describing the Ising type spin-spin interaction in the optical system

$$H_I = \sum_{j=1}^{N} J_z \sigma_j^z \sigma_{j+1}^z, \tag{6}$$

where $J_z = \sum_k \frac{g^2 e^{ik}}{2\delta_k}$. The parameters J_z can be tuned by varying coupling strength g and detuning δ_k. The evolution operator for the system is given by

$$U(t) = e^{-iH_I t} = e^{-i[\sum_{j=1}^{N} J_z \sigma_j^z \sigma_{j+1}^z]t}. \tag{7}$$

It has been shown that Ising interaction can be utilized to generate a new class of multipartite entanglement, the so called cluster states[13]. In the following discussions we will use this Hamiltonian to produce the cluster states.

In order to reduce notations we will not use the Ising interaction in Eq. (6) but rather the phase gate

$$U_p(t) = e^{-iH_I' t} \quad \text{with} \quad H_I' = -\sum_{j=1}^{N} J_z t \frac{1+\sigma_j^z}{2} \frac{1-\sigma_{j+1}^z}{2} \tag{8}$$

as the elementary two-qubit interaction between neighboring atoms[13, 14]. The equivalence between the quantum Ising interaction H_I in Eq. (6) and H_I' in Eq. (8) can be seen from the following discussion. From

$$\begin{aligned} H_I' = & -\sum_{j=1}^{N} J_z t \frac{1+\sigma_j^z}{2} \frac{1-\sigma_{j+1}^z}{2} \\ = & -\frac{1}{4} \sum_{j=1}^{N} J_z t (1 + \sigma_j^z - \sigma_{j+1}^z - \sigma_j^z \sigma_{j+1}^z) \end{aligned} \tag{9}$$

we find

$$\begin{aligned} U_p(t) = & \; e^{\frac{i}{4}\sum_{j=1}^{N} J_z t} e^{\frac{i}{4}\sum_{j=1}^{N} J_z t \sigma_j^z} \\ & \times e^{-\frac{i}{4}\sum_{j=1}^{N} J_z t \sigma_{j+1}^z} e^{-\frac{i}{4}\sum_{j=1}^{N} J_z t \sigma_j^z \sigma_{j+1}^z}. \end{aligned} \tag{10}$$

Therefore, the phase gate corresponds to the Ising interaction up to some additional $\frac{J_z t}{4}$-rotations around the z-axes at each qubit[14]. The entanglement properties are determined by the pure Ising interaction, which are not changed by the z-rotations.

We first consider the case of two coupled cavities. We denote $|+_j\rangle \equiv \frac{1}{\sqrt{2}}(|\downarrow_j\rangle + |\uparrow_j\rangle) = |g_j\rangle$, and $|-_j\rangle \equiv \frac{1}{\sqrt{2}}(|\downarrow_j\rangle - |\uparrow_j\rangle) = |e_j\rangle (j = 1, 2)$ for atomic states. Assume that initially the

atom in each cavity is prepared in state $|+_j\rangle = |g_j\rangle$. After an interaction time of $t = \pi/J_z$, we find that

$$U_p^{1,2}(\pi/J_z) = P_{z,-}^1 \otimes \mathbb{I}^2 + P_{z,+}^1 \otimes \sigma_2^z, \tag{11}$$

where $P_{z,\pm}^j = \frac{1\pm\sigma_j^z}{2}(j = 1, 2)$. Then the state evolution of the system is given by

$$\begin{aligned} U_p(\pi/J_z)|+_1\rangle|+_2\rangle &= \frac{1}{\sqrt{2}}(|\downarrow_1\rangle|-_2\rangle + |\uparrow_1\rangle|+_2\rangle) \\ &= \frac{1}{2}(|\downarrow_1\rangle\sigma_2^z + |\uparrow_1\rangle)(|\downarrow_2\rangle + |\uparrow_2\rangle). \end{aligned} \tag{12}$$

This state is a maximally entangled state. Up to a local unitary transformation on qubit 2, we can write it in the standard form.

We now turn to the case of many coupled microcavities. Initially all the atoms are prepared in states $|+_j\rangle = |g_j\rangle (j = 1, 2, ..., N)$. We choose $t = \pi/J_z$, then the produced state can be written in the following compact form[13, 14]

$$|\psi_N\rangle = \frac{1}{2^N}\bigotimes_{j=1}^{N}(|\downarrow_j\rangle\sigma_{j+1}^z + |\uparrow_j\rangle). \tag{13}$$

Then for $N = 3, 4$, one can obtain

$$\begin{aligned} |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|\downarrow_1\rangle|\downarrow_2\rangle|\downarrow_3\rangle + |\uparrow_1\rangle|\uparrow_2\rangle|\uparrow_3\rangle)_{l.u.} \\ |\psi_4\rangle &= \frac{1}{2}(|\downarrow_1\rangle|\downarrow_2\rangle|\downarrow_3\rangle|\downarrow_4\rangle + |\downarrow_1\rangle|\downarrow_2\rangle|\uparrow_3\rangle|\uparrow_4\rangle \\ &\quad +|\uparrow_1\rangle|\uparrow_2\rangle|\downarrow_3\rangle|\downarrow_4\rangle - |\uparrow_1\rangle|\uparrow_2\rangle|\uparrow_3\rangle|\uparrow_4\rangle)_{l.u.}. \end{aligned} \tag{14}$$

Where "l.u." indicates the equality holds up to a local unitary transformation on one or more of the qubits[13, 14]. $|\psi_3\rangle$ corresponds to a GHZ state of three qubits, but $|\psi_4\rangle$ is not equivalent to a 4-qubit GHZ state. The state $|\psi_4\rangle$ is the 4-qubit cluster state.

It is necessary to verify the approximations by numerics. We numerically simulate the dynamics generated by the full Hamiltonian H and compare it with the results generated by the effective model (6). As an example here we only consider the case of two atoms in two cavities. Initially the atoms stay in the state $|g\rangle_1|g\rangle_2$, and the cavity mode in the vacuum. In Fig. 2(a) we display the occupation probability $p_{g_1g_2}$ of system in the state $|g_1\rangle|g_2\rangle$, and the occupation of the photon number occupation $p_{N_1} = \langle\hat{a}_1^\dagger\hat{a}_1\rangle$ calculated both from

the full Hamiltonian and the effective model. Fig. 2(b) shows the von Neumann entropy of the reduced density matrix of one spin, i.e., $E_{vN} = -\mathrm{Tr}(\rho_1 \log_2 \rho_1(t))$. A maximally entangled state for the atoms occurs for $t = (2n+1)\pi/4J_z$ for any integer n. We choose the parameters as $\Omega = 50$ GHZ, $g = 0.1$ GHZ, $J_c = 0.02$ GHZ and $\omega_c - \omega_L = 1$ GHZ. It can be seen that the effective model can describe the dynamics very well provided that the parameters are appropriately chosen. The occupations of the photon number are always smaller than 0.01. Discrepancies between the numerical results for the full Hamiltonian and the effective Hamiltonian are due to the higher order terms for the detunings and Rabi frequencies. However, these discrepancies are below 2% with respect to the results from the full Hamiltonian.

We now consider some experimental matters. For experimental implementation, the parameter of the effective Hamiltonian J_z must be much larger than decay rates of cavity and atomic excited states. The cavity decay has neglectable effect on this scheme. We have only to consider the effect of the decay of atomic excited states. This proposal requires that the life time of the atomic excited states be longer than the time needed to complete the whole procedure. For potential atomic system, Rydberg atoms are good candidate. Promising candidates for microcavities are photonic bandgap cavities[11], and toroidal or spherical microcavities coupled via tapered optical fibers[10].

In summary, we have proposed a scheme for realizing the Ising spin-spin interaction and atomic cluster states utilizing trapped atoms in coupled microcavities. It is shown that the atoms can interact with each other via the exchange of virtual photons of the cavities. Through suitably tuning the parameters, an effective Ising spin-spin interaction can be generated in this optical system. We discuss how to use this Ising interaction to produce the cluster states of atoms. This scheme does not need the preparation of initial states of atoms and cavity modes, and is insensitive to cavity decay. With presently available experimental setups in cavity QED, it may be implemented.

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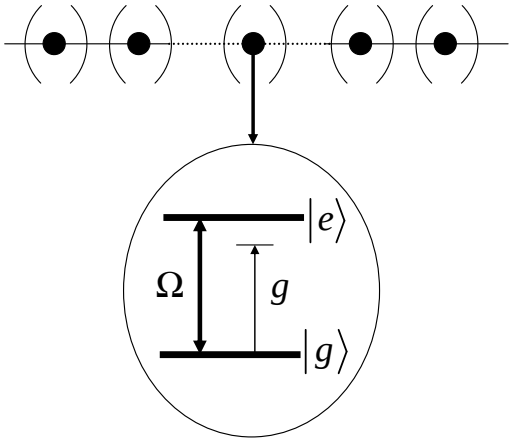

FIG. 1: An array of microcavities as described in the scheme. Photon hopping occurs because of the overlap of cavity modes of adjacent resonators. Two-level atom in each cavity is driven by external strong field.

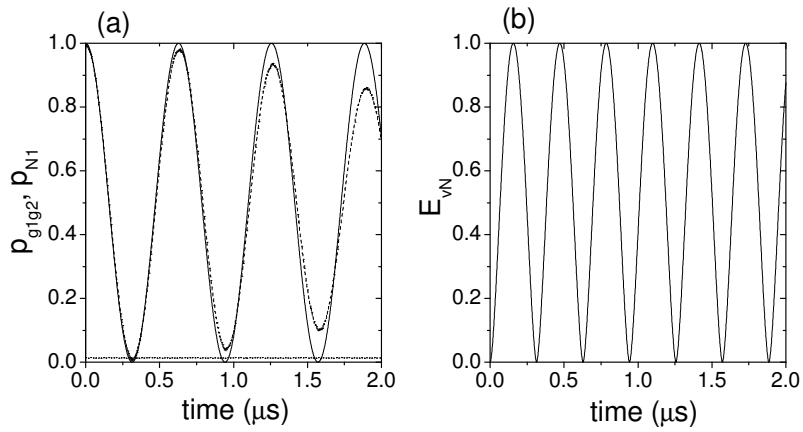

FIG. 2: (a) The occupation probability $p_{g_1 g_2}$ of the system in the state $|g_1\rangle|g_2\rangle$(solid line represents results from the effective model, and dash line represents results from the full Hamiltonian), and the photon number occupation $p_{N_1} = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle$. (b) The von Neumann entropy E_{vN} of the reduced density matrix of 1 effective spin. Parameters are chosen as $\Omega = 50$ GHZ, $g = 0.1$ GHZ, $J_c = 0.02$ GHZ and $\omega_c - \omega_L = 1$ GHZ.